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Some Remarks on the Boundedness and Convergence Properties of Smooth Sliding Mode Controllers

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Abstract: Conventional sliding mode controllers are based on the assumption of switching control, but a well-known drawback of such controllers is the chattering phenomenon. To overcome the undesirable chattering effects, the discontinuity in the control law can be smoothed out in a thin boundary layer neighboring the switching surface. In this paper, rigorous proofs of the boundedness and convergence properties of smooth sliding mode controllers are presented. This result corrects flawed conclusions previously reached in the literature. An illustrative example is also presented in order to confirm the convergence of the tracking error vector to the defined bounded region.

Keywords: Convergence analysis, Lyapunov methods, nonlinear control, sliding mode.

1 Introduction

The sliding mode control theory was conceived and developed in the former Soviet Union by Emelyanov and Kostyleva[1], Filippov[2], Itkis[3], Utkin[4], and others. However, a known drawback of conventional sliding mode controllers is the chattering phenomenon due to the discontinuous term in the control law. In order to avoid the undesired effects of the control chattering, Slotine[5−7] proposed the adoption of a thin boundary layer neighboring the switching surface, by replacing the sign function by a saturation function. This substitution can minimize or, when desired, even completely eliminate chattering, but turns perfect tracking into a tracking with guaranteed precision problem, which actually means that a steady-state error will always remain.

This paper presents a convergence analysis of smooth sliding mode controllers. The finite-time convergence of the tracking error vector to the boundary layer is handled using Lyapunov′s direct method. It is also analytically proven that, once in boundary layer, the tracking error vector exponentially converges to a bounded region. This result corrects a minor flaw in Slotine′s work, by showing that the bounds of the error vector are different from the bounds provided in [5 − 7]. Although the bounds proposed by Slotine are incorrect, they are until now widely evoked to establish the boundedness and convergence properties of many control schemes[8−14]. A simulation example is also presented in order to illustrate the convergence of the tracking error vector to the defined bounded region.

2 Problem statement

Consider a class of n-th order nonlinear systems

$$x^{(n)} = f(\boldsymbol{x}) + b(\boldsymbol{x})u \tag{1}$$

where u is the control input, the scalar variable x is the output of interest, $x^{(n)}$ is the n-th derivative of x with respect to time $t \in [0, \infty)$, $\boldsymbol{x} = [x, \dot{x}, \cdots, x^{(n-1)}]$ is the system state vector, and $f, b : \mathbf{R}^n \to \mathbf{R}$ are both nonlinear functions.

In respect of the dynamic system presented in (1), the following assumptions will be made.

Assumption 1. The function f is unknown but bounded by a known function of $\boldsymbol{x}$, i.e., $|\hat{f}(\boldsymbol{x}) - f(\boldsymbol{x})| \leqslant F(\boldsymbol{x})$, where $\hat{f}$ is an estimate of f.

Assumption 2. The input gain $b(\boldsymbol{x})$ is unknown but positive and bounded, i.e., $0 < b_{\min} \leqslant b(\boldsymbol{x}) \leqslant b_{\max}$.

The proposed control problem is to ensure that, even in the presence of parametric uncertainties and unmodeled dynamics, the state vector $\boldsymbol{x}$ will follow a desired trajectory $\boldsymbol{x}_d = [x_d, \dot{x}_d, \cdots, x_d^{(n-1)}]$ in the state space.

Regarding the development of the control law, the following assumptions should also be made.

Assumption 3. The state vector $\boldsymbol{x}$ is available.

Assumption 4. The desired trajectory $\boldsymbol{x}_d$ is once differentiable in time. Furthermore, every element of vector $\boldsymbol{x}_d$, as well as $x_d^{(n)}$, is available and with known bounds.

Now, let $\tilde{x} = x - x_d$ be defined as the tracking error in the variable x, and

$$\tilde{\boldsymbol{x}} = \boldsymbol{x} - \boldsymbol{x}_d = [\tilde{x}, \dot{\tilde{x}}, \cdots, \tilde{x}^{(n-1)}]$$

as the tracking error vector.

Consider a sliding surface S defined in the state space by the equation $s(\tilde{\boldsymbol{x}}) = 0$, with the function $s : \mathbf{R}^n \to \mathbf{R}$ satisfying

$$s(\tilde{\boldsymbol{x}}) = \left(\frac{\mathrm{d}}{\mathrm{d}t} + \lambda\right)^{n-1} \tilde{x}$$

conveniently rewritten as

$$s(\tilde{\boldsymbol{x}}) = \boldsymbol{c}^{\mathrm{T}} \tilde{\boldsymbol{x}} \tag{2}$$

where $\boldsymbol{c} = [c_{n-1}\lambda^{n-1}, \cdots, c_1\lambda, c_0]$, and c_i states for binomial coefficients, i.e.,

$$c_i = \binom{n-1}{i} = \frac{(n-1)!}{(n-i-1)!\, i!}\,, \quad i = 0, 1, \cdots, n-1 \tag{3}$$

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which makes $c_{n-1}\lambda^{n-1}+\cdots+c_1\lambda+c_0$ a Hurwitz polynomial.

From (3), it can be easily verified that $c_0 = 1$, for $\forall n \geqslant 1$. Thus, for notational convenience, the time derivative of s will be written in the following form:

$$\dot{s} = \boldsymbol{c}^{\mathrm{T}}\dot{\tilde{\boldsymbol{x}}} = \tilde{x}^{(n)} + \bar{\boldsymbol{c}}^{\mathrm{T}}\tilde{\boldsymbol{x}} \tag{4}$$

where $\bar{\boldsymbol{c}} = [0, c_{n-1}\lambda^{n-1}, \cdots, c_1\lambda]$.

Now, let the problem of controlling the uncertain nonlinear system (1) be treated via the classical sliding mode approach, defining a control law composed of an equivalent control $\hat{u} = \hat{b}^{-1}(-\hat{f} + x_d^{(n)} - \bar{\boldsymbol{c}}^{\mathrm{T}}\tilde{\boldsymbol{x}})$ and a discontinuous term $-K\mathrm{sgn}(s)$:

$$u = \hat{b}^{-1}\left(-\hat{f} + x_d^{(n)} - \bar{\boldsymbol{c}}^{\mathrm{T}}\tilde{\boldsymbol{x}}\right) - K\mathrm{sgn}(s) \tag{5}$$

where $\hat{b} = \sqrt{b_{\max}b_{\min}}$ is an estimate of b, K is a positive gain, and $\mathrm{sgn}(\cdot)$ is defined as

$$\mathrm{sgn}(s) = \begin{cases} -1, & \text{if} \quad s < 0 \\ 0, & \text{if} \quad s = 0 \\ 1, & \text{if} \quad s > 0. \end{cases}$$

Based on Assumptions 1 and 2 and considering that $\beta^{-1} \leqslant \hat{b}/b \leqslant \beta$, where $\beta = \sqrt{b_{\max}/b_{\min}}$, the gain K should be chosen according to

$$K \geqslant \beta\hat{b}^{-1}(\eta + F) + (\beta - 1)\left|\hat{b}^{-1}(-\hat{f} + x_d^{(n)} - \bar{\boldsymbol{c}}^{\mathrm{T}}\tilde{\boldsymbol{x}})\right| \tag{6}$$

where η is a strictly positive constant related to the reaching time.

Therefore, it can be easily verified that (5) is sufficient to impose the sliding condition

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}s^2 \leqslant -\eta|s|$$

which, in fact, ensures the finite-time convergence of the tracking error vector to the sliding surface S and, consequently, its exponential stability.

However, the presence of a discontinuous term in the control law leads to the well-known chattering effect. To avoid these undesirable high-frequency oscillations of the controlled variable, Slotine[5−7] proposed the adoption of a thin boundary layer, S_ϕ, in the neighborhood of the switching surface:

$$S_\phi = \left\{\tilde{\boldsymbol{x}} \in \mathbf{R}^n \;\middle|\; |s(\tilde{\boldsymbol{x}})| \leqslant \phi\right\} \tag{7}$$

where ϕ is a strictly positive constant that represents the boundary layer thickness.

The boundary layer is achieved by replacing the sign function by a continuous interpolation inside S_ϕ. It should be emphasized that this smooth approximation, which will be called here $\varphi(s, \phi)$, must behave exactly like the sign function outside the boundary layer. There are several options to smooth out the ideal relay but the most common choices are the saturation function:

$$\mathrm{sat}\left(\frac{s}{\phi}\right) = \begin{cases} \mathrm{sgn}(s), & \text{if} \quad \left|\dfrac{s}{\phi}\right| \geqslant 1 \\[2ex] \dfrac{s}{\phi}, & \text{if} \quad \left|\dfrac{s}{\phi}\right| < 1 \end{cases} \tag{8}$$

and the hyperbolic tangent function $\tanh(s/\phi)$.

In this way, the smooth sliding mode control law can be stated as follows:

$$u = \hat{b}^{-1}\left(-\hat{f} + x_d^{(n)} - \bar{\boldsymbol{c}}^{\mathrm{T}}\tilde{\boldsymbol{x}}\right) - K\varphi(s, \phi). \tag{9}$$

3 Convergence analysis

The attractiveness and invariant properties of the boundary layer are established in the following theorem.

Theorem 1. Consider the uncertain nonlinear system (1) and Assumptions 1–4. Then, the smooth sliding mode controller defined by (6) and (9) ensures the finite-time convergence of the tracking error vector to the boundary layer S_ϕ, defined according to (7).

Proof. Let a positive-definite Lyapunov function candidate V be defined as

$$V(t) = \frac{1}{2}s_\phi^2$$

where s_ϕ is a measure of the distance of the current error to the boundary layer, and can be computed as follows:

$$s_\phi = s - \phi\,\mathrm{sat}\left(\frac{s}{\phi}\right). \tag{10}$$

Noting that $s_\phi = 0$ in the boundary layer, one has $\dot{V}(t) = 0$ inside S_ϕ. From (8) and (10), it can be easily verified that $\dot{s}_\phi = \dot{s}$ outside the boundary layer and, in this case, $\dot{V}$ becomes

$$\begin{aligned}\dot{V}(t) =& s_\phi\dot{s}_\phi = s_\phi\dot{s} = (x^{(n)} - x_d^{(n)} + \bar{\boldsymbol{c}}^{\mathrm{T}}\tilde{\boldsymbol{x}})s_\phi = \\ & \left(f + bu - x_d^{(n)} + \bar{\boldsymbol{c}}^{\mathrm{T}}\tilde{\boldsymbol{x}}\right)s_\phi.\end{aligned}$$

Considering that outside the boundary layer, the control law (9) takes the following form:

$$u = \hat{b}^{-1}\left(-\hat{f} + x_d^{(n)} - \bar{\boldsymbol{c}}^{\mathrm{T}}\tilde{\boldsymbol{x}}\right) - K\mathrm{sgn}(s_\phi)$$

and noting that $f = \hat{f} - (\hat{f} - f)$, we have

$$\begin{aligned}\dot{V}(t) =& \big[f + b\hat{b}^{-1}(-\hat{f} + x_d^{(n)} - \bar{\boldsymbol{c}}^{\mathrm{T}}\tilde{\boldsymbol{x}}) - \\ & bK\mathrm{sgn}(s_\phi) - x_d^{(n)} + \bar{\boldsymbol{c}}^{\mathrm{T}}\tilde{\boldsymbol{x}}\big]s_\phi = \\ & -\big[(\hat{f} - f) - b\hat{b}^{-1}(-\hat{f} + x_d^{(n)} - \bar{\boldsymbol{c}}^{\mathrm{T}}\tilde{\boldsymbol{x}}) + \\ & (-\hat{f} + x_d^{(n)} - \bar{\boldsymbol{c}}^{\mathrm{T}}\tilde{\boldsymbol{x}}) + bK\mathrm{sgn}(s_\phi)\big]s_\phi.\end{aligned}$$

Therefore, considering Assumptions 1 and 2 and defining K according to (6), $\dot{V}$ becomes

$$\dot{V}(t) \leqslant -\eta|s_\phi|$$

which implies $V(t) \leqslant V(0)$ and that s_ϕ is bounded. From the definition of s and s_ϕ, respectively (2) and (10), it can be verified that $\tilde{\boldsymbol{x}}$ is bounded. Thus, Assumption 4 and (4) imply that $\dot{s}$ is also bounded.

The finite-time convergence of the tracking error vector to the boundary layer can be shown by recalling that

$$\dot{V}(t) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}s_\phi^2 = s_\phi\dot{s}_\phi \leqslant -\eta|s_\phi|.$$

Then, dividing by $|s_\phi|$ and integrating both sides between 0 and t gives

$$\int_0^t \frac{s_\phi}{|s_\phi|}\dot{s}_\phi\,\mathrm{d}\tau \leqslant -\int_0^t \eta\,\mathrm{d}\tau$$

$$|s_\phi(t)| - |s_\phi(0)| \leqslant -\eta\, t.$$

In this way, considering $t_{\rm reach}$ as the time required to hit S_ϕ and noting that $|s_\phi(t_{\rm reach})| = 0$, one has

$$t_{\rm reach} \leqslant \frac{|s_\phi(0)|}{\eta}$$

which guarantees the convergence of the tracking error vector to the boundary layer in a time interval smaller than $|s_\phi(0)|/\eta$. □

Therefore, the value of the positive constant η can be properly chosen in order to keep the reaching time, $t_{\rm reach}$, as short as possible. Fig. 1 shows that the time evolution of $|s_\phi|$ is bounded by the straight line $|s_\phi(t)| = |s_\phi(0)| - \eta\, t$.

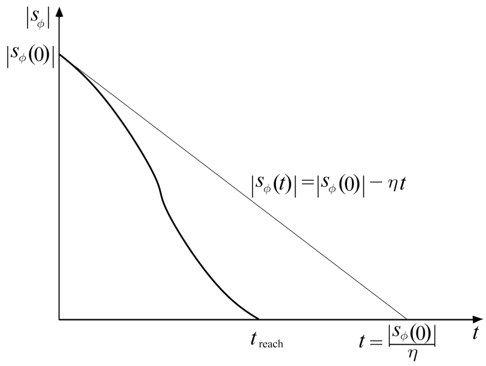

Fig. 1 Time evolution of $|s_\phi|$

Finally, the proof of the boundedness of the tracking error vector relies on Theorem 2.

Theorem 2. Let the boundary layer S_ϕ be defined according to (7). Then, once inside S_ϕ, the tracking error vector will exponentially converge to an n-dimensional box defined according to $|\tilde{x}^{(i)}| \leqslant \zeta_i \lambda^{i-n+1}\phi, i = 0, 1, \cdots, n-1\}$, with ζ_i satisfying

$$\zeta_i = \begin{cases} 1, & \text{for} \quad i = 0 \\ 1 + \sum\limits_{j=0}^{i-1} \binom{i}{j}\zeta_j, & \text{for} \quad i = 1, 2, \cdots, n-1. \end{cases} \tag{11}$$

Proof. From the definition of s, (2), and considering that $|s(\boldsymbol{x})| \leqslant \phi$ may be rewritten as $-\phi \leqslant s(\boldsymbol{x}) \leqslant \phi$, we have

$$-\phi \leqslant c_0\tilde{x}^{(n-1)} + c_1\lambda\tilde{x}^{(n-2)} + \cdots + c_{n-1}\lambda^{n-1}\tilde{x} \leqslant \phi. \tag{12}$$

Multiplying (12) by $\mathrm{e}^{\lambda t}$ yields

$$-\phi\mathrm{e}^{\lambda t} \leqslant \frac{\mathrm{d}^{n-1}}{\mathrm{d}t^{n-1}}(\tilde{x}\mathrm{e}^{\lambda t}) \leqslant \phi\mathrm{e}^{\lambda t} \tag{13}$$

Integrating (13) between 0 and t gives

$$-\frac{\phi}{\lambda}\mathrm{e}^{\lambda t} + \frac{\phi}{\lambda} \leqslant \frac{\mathrm{d}^{n-2}}{\mathrm{d}t^{n-2}}(\tilde{x}\mathrm{e}^{\lambda t}) - \left.\frac{\mathrm{d}^{n-2}}{\mathrm{d}t^{n-2}}(\tilde{x}\mathrm{e}^{\lambda t})\right|_{t=0} \leqslant \frac{\phi}{\lambda}\mathrm{e}^{\lambda t} - \frac{\phi}{\lambda}. \tag{14}$$

(14) can be conveniently rewritten as

$$-\frac{\phi}{\lambda}\mathrm{e}^{\lambda t} - \left(\left|\left.\frac{\mathrm{d}^{n-2}}{\mathrm{d}t^{n-2}}(\tilde{x}\mathrm{e}^{\lambda t})\right|_{t=0}\right| + \frac{\phi}{\lambda}\right) \leqslant \frac{\mathrm{d}^{n-2}}{\mathrm{d}t^{n-2}}(\tilde{x}\mathrm{e}^{\lambda t}) \leqslant \frac{\phi}{\lambda}\mathrm{e}^{\lambda t} + \left(\left|\left.\frac{\mathrm{d}^{n-2}}{\mathrm{d}t^{n-2}}(\tilde{x}\mathrm{e}^{\lambda t})\right|_{t=0}\right| + \frac{\phi}{\lambda}\right). \tag{15}$$

The same reasoning can be repeatedly applied until the $(n-1)$-th integral of (13) is reached:

$$-\frac{\phi}{\lambda^{n-1}}\mathrm{e}^{\lambda t} - \left(\left|\left.\frac{\mathrm{d}^{n-2}}{\mathrm{d}t^{n-2}}(\tilde{x}\mathrm{e}^{\lambda t})\right|_{t=0}\right| + \frac{\phi}{\lambda}\right)\frac{t^{n-2}}{(n-2)!} - \cdots - \left(|\tilde{x}(0)| + \frac{\phi}{\lambda^{n-1}}\right) \leqslant \tilde{x}\mathrm{e}^{\lambda t} \leqslant \frac{\phi}{\lambda^{n-1}}\mathrm{e}^{\lambda t} + \left(\left|\left.\frac{\mathrm{d}^{n-2}}{\mathrm{d}t^{n-2}}(\tilde{x}\mathrm{e}^{\lambda t})\right|_{t=0}\right| + \frac{\phi}{\lambda}\right)\frac{t^{n-2}}{(n-2)!} + \cdots + \left(|\tilde{x}(0)| + \frac{\phi}{\lambda^{n-1}}\right) \tag{16}$$

Furthermore, dividing (16) by $\mathrm{e}^{\lambda t}$, it can be easily verified that, for $t \to \infty$,

$$-\frac{\phi}{\lambda^{n-1}} \leqslant \tilde{x}(t) \leqslant \frac{\phi}{\lambda^{n-1}} \tag{17}$$

Considering the $(n-2)$-th integral of (13)

$$-\frac{\phi}{\lambda^{n-2}}\mathrm{e}^{\lambda t} - \left(\left|\left.\frac{\mathrm{d}^{n-2}}{\mathrm{d}t^{n-2}}(\tilde{x}\mathrm{e}^{\lambda t})\right|_{t=0}\right| + \frac{\phi}{\lambda}\right)\frac{t^{n-3}}{(n-3)!} - \cdots - \left(|\dot{\tilde{x}}(0)| + \frac{\phi}{\lambda^{n-2}}\right) \leqslant \frac{\mathrm{d}}{\mathrm{d}t}(\tilde{x}\mathrm{e}^{\lambda t}) \leqslant \frac{\phi}{\lambda^{n-2}}\mathrm{e}^{\lambda t} + \left(\left|\left.\frac{\mathrm{d}^{n-2}}{\mathrm{d}t^{n-2}}(\tilde{x}\mathrm{e}^{\lambda t})\right|_{t=0}\right| + \frac{\phi}{\lambda}\right)\frac{t^{n-3}}{(n-3)!} + \cdots + \left(|\dot{\tilde{x}}(0)| + \frac{\phi}{\lambda^{n-2}}\right) \tag{18}$$

and noting that $\mathrm{d}(\tilde{x}\mathrm{e}^{\lambda t})/\mathrm{d}t = \dot{\tilde{x}}\mathrm{e}^{\lambda t} + \tilde{x}\lambda\mathrm{e}^{\lambda t}$, by imposing the bounds (17) to (18) and dividing it again by $\mathrm{e}^{\lambda t}$ for $t \to \infty$, we have

$$-2\frac{\phi}{\lambda^{n-2}} \leqslant \dot{\tilde{x}}(t) \leqslant 2\frac{\phi}{\lambda^{n-2}}. \tag{19}$$

Now, applying the bounds (17) and (19) to the $(n-3)$-th integral of (13) and dividing it once again by $\mathrm{e}^{\lambda t}$ for $t \to \infty$, we have

$$-6\frac{\phi}{\lambda^{n-3}} \leqslant \ddot{\tilde{x}}(t) \leqslant 6\frac{\phi}{\lambda^{n-3}}. \tag{20}$$

The same procedure can be successively repeated until the bounds for $\tilde{x}^{(n-1)}$ are achieved:

$$-\left(1 + \sum_{i=0}^{n-2}\binom{n-1}{i}\zeta_i\right)\phi \leqslant \tilde{x}^{(n-1)} \leqslant \left(1 + \sum_{i=0}^{n-2}\binom{n-1}{i}\zeta_i\right)\phi \tag{21}$$

where the coefficients ζ_i $(i = 0, 1, \cdots, n-2)$ are related to the previously obtained bounds of each $\tilde{x}^{(i)}$ and can be summarized as in (11).

In this way, by inspection of the integrals of (13), as well as (17), (19)–(21) and the other omitted bounds, the tracking error will be confined within the limits $|\tilde{x}^{(i)}| \leqslant \zeta_i\lambda^{i-n+1}\phi$, $i = 0, 1, \cdots, n-1$, where ζ_i is defined by (11). □

Remark 1. Theorem 2 corrects a minor error in [5 − 7]. Slotine proposed that the bounds for $\tilde{x}^{(i)}$ could be summarized as $|\tilde{x}^{(i)}| \leqslant 2^i\lambda^{i-n+1}\phi$, $i = 0, 1, \cdots, n-1$. Although both results lead to the same bounds for $\tilde{x}$ and $\dot{\tilde{x}}$, they start to differ from each other when the order of the derivative is higher than one, $i > 1$. For example, according to the method of Slotine, the bounds for the second derivative would be $|\ddot{\tilde{x}}| \leqslant 4\phi\lambda^{3-n}$ not $|\ddot{\tilde{x}}| \leqslant 6\phi\lambda^{3-n}$, as demonstrated in Theorem 2.

Remark 2. It must be noted that the n-dimensional box defined according to the aforementioned bounds is not completely inside the boundary layer. Considering the attractiveness and invariant properties of S_ϕ demonstrated in Theorem 1, the region of convergence can be stated as the intersection of the boundary layer and the n-dimensional box defined in Theorem 2. Therefore, the tracking error vector will exponentially converge to a closed region $\Phi = \{\boldsymbol{x} \in \mathbf{R}^n \mid |s(\tilde{\boldsymbol{x}})| \leqslant \phi \text{ and } |\tilde{x}^{(i)}| \leqslant \zeta_i \lambda^{i-n+1}\phi,\ i = 0, 1, \cdots, n-1\}$, with ζ_i defined by (11).

Fig. 2 illustrates the region of convergence Φ, defined according to Remark 2, for a second-order system ($n = 2$).

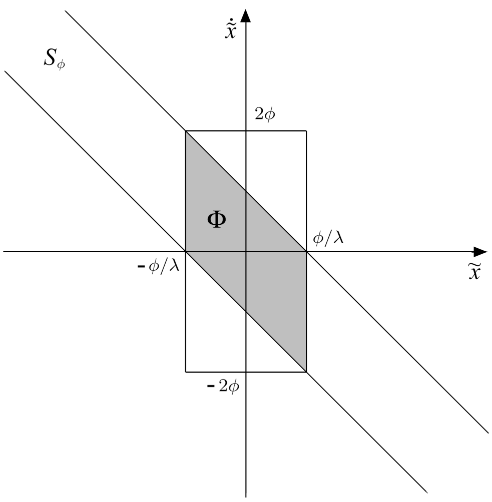

Fig. 2 Bounds of $x^{(i)}$ for a second-order system

4 Illustrative example

In order to confirm the convergence of the tracking error vector to the bounded region defined in Remark 2, consider a controlled Van der Pol oscillator

$$\ddot{x} - \mu(1 - x^2)\dot{x} + x = bv \tag{22}$$

with a dead-band in the control input defined according to

$$v = \begin{cases} u + 0.4, & \text{if} \quad u \leqslant -0.4 \\ 0, & \text{if} \quad -0.4 < u < 0.4 \\ u - 0.4, & \text{if} \quad u \geqslant 0.4. \end{cases} \tag{23}$$

The unforced Van der Pol oscillator, i.e., by considering $u = 0$, exhibits a limit cycle. The control objective is to let the state vector $\boldsymbol{x} = [x, \dot{x}]$ track a desired trajectory $\boldsymbol{x}_d = [\sin t, \cos t]$ situated inside the limit cycle. Fig. 3 shows the phase portrait of the unforced Van der Pol oscillator with the limit cycle, two convergent orbits and the desired trajectory.

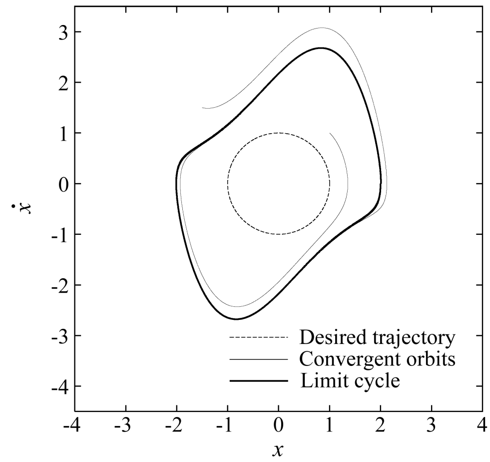

Fig. 3 Phase portrait of the unforced Van der Pol oscillator

The simulation study was performed with an implementation in C, with sampling rates of 500 Hz for control system and 1 kHz for the Van der Pol oscillator, and the differential equations were numerically solved using the fourth-order Runge-Kutta method. The chosen parameters for the Van der Pol oscillator were $b = 1$ and $\mu = 1$.

Regarding the controller design, to ratify its robustness against both structured and unstructured uncertainties, an uncertainty of $\pm 20\%$ over the value of b was taken into account, i.e., $b_{\min} = 0.8$ and $b_{\max} = 1.2$, and the dead-band was treated as modeling imprecision, i.e., not considered in controller design. In this way, for a second-order system with state vector $\boldsymbol{x} = [x, \dot{x}]$ and $s = \dot{\tilde{x}} + \lambda\tilde{x}$, a smooth sliding mode controller can be chosen as follows:

$$u = \hat{b}^{-1}[-\mu(1 - x^2)\dot{x} + x + \ddot{x}_d - \lambda\dot{\tilde{x}}] - K\text{sat}\left(\frac{s}{\phi}\right).$$

The following parameters were adopted for the controller: $\beta = 1.22$, $\hat{b} = 0.98$, $\eta = 0.1$, $\lambda = 0.6$, $\phi = 0.1$, and $F = 0.32$.

Considering that the initial state and initial desired state are not equal, $\tilde{\boldsymbol{x}}(0) = [-2.0, -0.4]$, Figs. 4–6 show the corresponding results for the tracking of $\boldsymbol{x}_d = [\sin t, \cos t]$.

As observed in Fig. 6, the tracking error vector is driven to the proposed region of convergence and remains inside Φ as $t \to \infty$, even in the presence of modeling imprecisions.

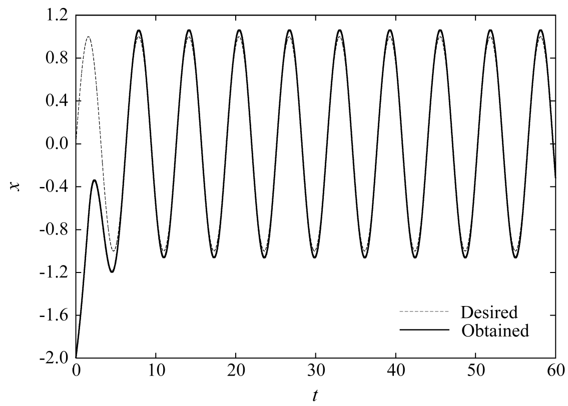

Fig. 4 Tracking performance

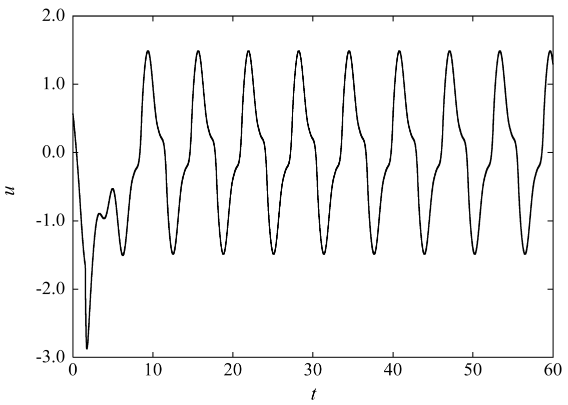

Fig. 5 Control action

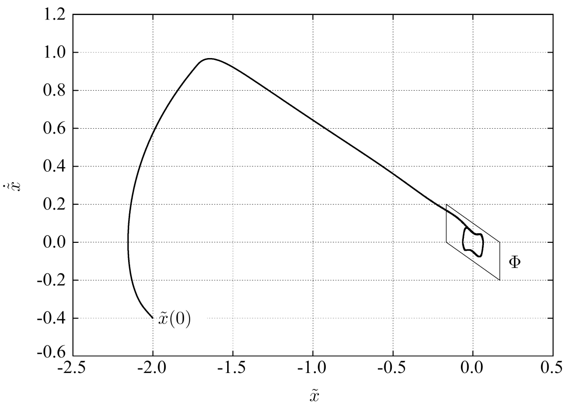

Fig. 6 Phase portrait of the tracking

5 Conclusions

In this paper, a convergence analysis of smooth sliding mode controllers was presented. The attractiveness and invariant properties of the boundary layer as well as the exponential convergence of the tracking error vector to a bounded region were analytically proven. This last result corrected flawed conclusions previously reached in the literature. Numerical simulations with a Van der Pol oscillator confirm the convergence of the tracking error vector to the proposed region of convergence.

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